Assignment 7

This homework is due *Friday*, November 11.

There are total 29 points in this assignment. 26 points is considered 100%. If you go over 26 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper. Your solutions should contain full proofs/calculations (unless stated otherwise in the problem). Bare answers will not earn you much.

This assignment covers sections 4.3–4.5 in O'Neill.

(1) (4.3.6) Let $\mathbf{x}$ and $\mathbf{y}$ be patches in the unit sphere Σ that are defined on the unit disk $D : u^2 + v^2 < 1$ by

$$\mathbf{x}(u, v) = (u, v, f(u, v)), \quad \mathbf{y}(u, v) = (v, f(u, v), u),$$

where $f = \sqrt{1 - u^2 - v^2}$.

(a) [2pt] On a sketch of Σ indicate the images $\mathbf{x}(D)$ and $\mathbf{y}(D)$, and the region on which the overlap.

(b) [3pt] At which points of D is $\mathbf{y}^{-1}\mathbf{x}$ is defined? Find a formula for this function.

(c) [3pt] At which points of D is $\mathbf{x}^{-1}\mathbf{y}$ is defined? Find a formula for this function.

(2) [3pt] (4.4.1) Prove the Leibnizian formulas

$$d(fg) = gdf + fdg, \quad d(f\varphi) = df \wedge \varphi + fd\varphi,$$

where f and g are functions on a surface M and φ is a 1-form. (*Hint:* By definition, $(f\varphi)(v_p) = f(p)\varphi(v_p)$. Hence, $f\varphi$ evaluated on $\mathbf{x}_u$ is $f(\mathbf{x})\varphi(\mathbf{x}_u)$.)

(3) [3pt] (4.4.4c) If f, g are functions on a surface M, prove that

$$(df \wedge dg)(v, w) = v[f]w[g] - v[g]w[f].$$

(4) [2pt] If Σ is the sphere $\|p\| = 1$, the mapping $A : \Sigma \to \Sigma$ such that $A(p) = -p$ is called the *antipodal map* of Σ. Prove that A is a diffeomorphism and $A*(v_p) = (-v)_{-p}$.

(5) (4.5.10) Given mappings $F : M \to N$, $G : N \to P$, let $GF : M \to P$ be the composite mapping. Show that

(a) [3pt] GF is differentiable,

(b) [3pt] $(GF)* = G*F*$,

(c) [3pt] $(GF)^* = F^*G^*$, i.e. for any form ξ on P, $(GF)^*(\xi) = F^*(G^*(\xi))$.

— *see next page* —

(6) (4.5.12)

(a) [2pt] Show that the inverse mapping P^{-1} of the stereographic projection $P : \Sigma \to \mathbb{R}^2$ is given by

$$P^{-1}(u, v) = \frac{(4u, 4v, 2f)}{f + 4}, \quad \text{where } f = u^2 + v^2.$$

(Check that both PP^{-1} and $P^{-1}P$ are identity maps.)

(b) [2pt] Deduce that the entire sphere Σ can be covered by only two patches.